

Non-Abelian Vortex-String Dynamics from Nonlinear Realization

Lu-Xin Liu^a and Muneto Nitta^b

a. Department of Physics,

Purdue University,

West Lafayette, IN 47907, USA

liul@physics.purdue.edu

b. Department of Physics, and

Research and Education Center for Natural Sciences

Keio University,

4-1-1 Hiyoshi, Yokohoma, Kanagawa, 223-8521, Japan

nitta@phys-h.keio.ac.jp

Abstract. The dynamics of the non-Abelian vortex-string, which describes its low energy oscillations into the target $D = 3 + 1$ spacetime as well as its orientations in the internal space, is derived by the approach of nonlinear realization. The resulting action correlating these two sectors is found to have an invariant synthesis form of the Nambu-Goto- CP^{N-1} model actions. Higher order corrections to the vortex actions are presented up to the order of quartic derivatives. General p -brane dynamics in terms of the internal symmetry breaking is also discussed.

1 Introduction

It is well known that topological defects as physical objects can form in systems that exhibit the phenomenon of spontaneous symmetry breaking. In fact, as early as in the early seventies, the spontaneously broken gauge theory was found to possess these soliton-like solutions, namely, the Nielsen-Olesen vortex-string [1] and the magnetic monopole of 't Hooft and Polyakov [2], corresponding to Abelian and non-Abelian gauge theories, respectively.

The vortex-string topological defect, on which particular attention has been focused, plays important roles in diverse areas of physics, covering condensed matter, particle physics as well as cosmology [3]. In addition, in brane world scenarios, vortices as codimension two branes are deemed to be especially useful when considering localization of gauge fields by using warped compactifications [4].

Actually the vortex-string topological defect, such as the Nielsen-Olesen vortex line, can form if the vacuum manifold M is not simply connected, $\pi_1(M) \neq 0$. The vacuum manifold is $M = U(1)$ in the Abelian Higgs model, admitting the Nielsen-Olesen vortex line [1]. On the other hand, the vortex-strings arising from non-Abelian gauge theories have attracted much attention and been extensively studied in supersymmetric gauge theories [5]–[11], see [12] as a review. There, the underlying theory is a $U(N_C)$ gauge theory with $N_F(\geq N_C)$ Higgs fields in the fundamental representation with the common $U(1)$ charges. In the case when $N_F = N_C \equiv N$, the theory is found to have a unique vacuum state, i.e. the so called color-flavor locking phase, whose vacuum state preserves a global unbroken $SU(N)_{C+F}$ symmetry. Such a system admits the solution of Nielsen-Olesen type of vortices, and the solution further breaks this locking symmetry to $SU(N-1) \times U(1)$. Therefore, unlike the Abelian vortices, these non-Abelian vortices have moduli (zero modes) corresponding to spontaneous breaking of the non-Abelian symmetry; i.e., besides the usual position moduli they also carry orientational moduli $CP^{N-1} = SU(N)/[SU(N-1) \times U(1)]$ in the color and flavor internal space.

Similar non-Abelian vortex-strings appear in QCD with high baryon density at low temperature [13] where the $SU(3)_C$ gauge symmetry is broken by diquark condensations and is locked with the $SU(3)_F$ flavor symmetry acting on the light quarks. Those vortices carry the CP^2 orientational zero modes [14, 15] and may

exist in the core of a neutron star [16].

The dynamics of such non-Abelian vortices, whose underlying theory illustrates both the spacetime and internal symmetry breaking, can be described by the collective coordinates in such an enlarged moduli space. Each moduli parameter provides a massless field for the effective field theory on the string world-sheet, which actually corresponds to the Nambu-Goldstone scalar associated with the spontaneously broken global symmetry.

As for the spontaneous symmetry breaking, the approach of nonlinear realization has been demonstrated a natural, economical and elegant framework for treating it. In fact, this method has been applied to a wide range of physical problems most notably in the form of nonlinear sigma (NLS) models [17], supersymmetry [18]–[20], brane theories [21]–[26], and combination of them. There, the Lagrangian is invariant with respect to the transformations of some continuous group G , but the ground state is not an invariant of G but only of some subgroup H . In this context, the resulting phenomenological Lagrangian becomes an effective theory at energies far below the scale of spontaneous symmetry breaking. Consequently, the effective action can be expressed in terms of the dynamics of these Nambu-Goldstone fields.

In the present context, the formation of the non-Abelian vortex breaks the spacetime and internal space symmetries at the same time, i.e., it breaks the target four-dimensional Minkowski spacetime to the lower two-dimensional world-sheet spacetime; meanwhile, it also has dynamical modes describing its orientations in the special color-flavor locked phase. The interactions between these two different sectors would be of interest to the general vortex-string theory and is worthy of detail exploration and investigation. Beside, as the string is embedded in a higher target spacetime, the terms that characterize this embedding or represent the string rigidity may also be supplemented to its actions [27, 28].

The organization of this paper is as follows. In section 2, the spacetime symmetry breaking of the vortex-string is constructed in terms of the coset structure $ISO(1, 3)/[SO(1, 3) \times SO(2)]$. Then after considering the spontaneous breaking of the internal space, the kinetic terms associated with the Nambu-Goldstone fields are shown to be described by the metric on the internal coset manifold $CP^{N-1} = SU(N)/[SU(N-1) \times U(1)]$. Therefore, the low energy effective actions of

the vortex, which illustrate both the spacetime and internal space symmetry breakings, are obtained by means of Maurer-Cartan one-forms. As a result, in addition to the long wave oscillating modes associated with the translational directions transverse to the vortex-string, the string also has oscillating modes corresponding to its orientations in the internal space. The effective action between these two sectors is found to have a factorized form of the Nambu-Goto- \mathbf{CP}^{N-1} model actions. Besides, the term that describes the world-sheet embedding is given by the extrinsic curvature couplings and produces interactions containing quartic derivatives. In section 3, the general formalism of the p -brane dynamics corresponding to both spacetime and internal space symmetry breakings is constructed. The effective action is then has a synthesis form of the Nambu-Goto-NLS model actions. Section 4 is devoted to conclusion and discussion. In appendix A we give a relation with the Skyrme-Faddeev model for $N = 1$ (\mathbf{CP}^1) with 3+1 dimensional world-volume ($D = 5 + 1, p = 3$).

2 Effective Actions of Non-Abelian Vortex-Strings

Let us consider the non-Abelian vortex-strings described in Ref. [5, 6]. There, the Lagrangian has a $U(N)_C$ color symmetry along with an $SU(N)_F$ flavor symmetry. As a result of the overall $U(1)$ gauge symmetry breaking, it ensures the existence of the vortex solution for the underlying theory. In addition, the vacuum is found to remain a diagonal global color-flavor locking phase with $SU(N)_{C+F}$ symmetry. Furthermore, the vortex-string solution is found to further break this symmetry down to the $SU(N-1) \times U(1)$ symmetry. Since the diagonal color-flavor symmetry is not broken by the VEV of the scalar fields in the bulk, this breaking is physical and has nothing to do the Higgs mechanism caused by the gauge transformation. This fact therefore leads to the internal orientation moduli space of the string, and the presence of these modes makes the string genuinely non-Abelian. The whole moduli space of the non-Abelian string then has the form $\mathbf{C} \times \mathbf{CP}^{N-1}$, where \mathbf{C} is related to the coset space $ISO(1, 3)/[SO(1, 3) \times SO(2)]$, describing the transverse oscillations of the string in the translational directions of the world-sheet; while the internal degrees of freedom is given by the coset space $\mathbf{CP}^{N-1} \simeq SU(N)/[SU(N-1) \times U(1)]$,

corresponding to orientational moduli of the string in the internal color-flavor $SU(N)$ space.

We start first with the dynamics of the string corresponding to the spontaneous breaking of the spacetime symmetry in the presence of a vortex-string [21]. Place such a vortex-string along the x^1 -axis, the world-sheet is then parameterized by $\{x^0, x^1\}$ in the static gauge, while the target spacetime parameterized by the coordinates x^μ with $\mu = 0, 1, 2, 3$. Therefore, the stability subgroup H_o of the target spacetime symmetry breaking is given by the direct product of $SO(1, 1) \times SO(2)$, which corresponds to the Lorentz boost $SO(1, 1)$ symmetry (formed by the generator $M = M^{ab}$, $a, b = 0, 1$) and the rotational $SO(2)$ invariance (formed by the generator $T = M^{23}$) in the x^2 - x^3 plane respectively. Accordingly, the coset representative elements $\Omega_o = G_o/H_o = G_o/[SO(1, 1) \times SO(2)]$ can be exponentially parameterized as

$$\Omega_o = e^{ix^a P_a} e^{i\phi_i(x) Z_i} e^{iu_i^a(x) K_{ia}} \quad (2.1)$$

in the static gauge; in which x^a, ϕ_i, u_i^a are the collective coordinates parameterizing the coset space Ω_o , and G_o is the target $D = 3 + 1$ Poincare group. The broken generators are the automorphism generators $K_1^a = M^{a2}$ and $K_2^a = M^{a3}$ along with the broken spacetime generators Z_i ($Z_i = Z_1, Z_2 = P_2, P_3$) associated with the translational directions transverse to the string. We take the convention $\eta^{ab} = (+, -)$ in what follows of the paper.

The effective action of the string that describes its low energy oscillations into the covolume space can be constructed by using the Zweibein from the coset structure $\Omega_o^{-1} d\Omega_o$, which has an explicit expansion with respect to the G_o generators

$$\Omega_o^{-1} d\Omega_o = i(\omega^A P_A + \omega_{z_i} Z_i + \omega_{k_i}^A K_{iA} + \omega_T T + \omega_M M). \quad (2.2)$$

It is found that

$$\begin{aligned} i\omega^A P_A &= id x^B P_A (\delta_B^A + u_{iB} (U^{-1} (\cosh \sqrt{U} - 1))_{ij} u_j^A) \\ &\quad + id \phi_i (U^{-\frac{1}{2}} \sinh \sqrt{U})_{ij} u_j^A P_A \end{aligned} \quad (2.3)$$

where $A, B = 0, 1$ and $U_{ij} = u_i^a u_{ja}$. The capital letters A, B are used to represent the covariant spacetime coordinate indices of the string world-sheet, and the lowercase

letters a, b are used to represent $1 + 1$ general coordinate indices in what follows. Considering $\omega^A = dx^b e_{ob}^A$ in static gauge, the Zweibein is therefore found to have the form

$$e_{oa}^A = \delta_a^A + u_{ia}(U^{-1}(\cosh \sqrt{U} - 1))_{ij} u_j^A + \partial_a \phi_i (U^{-1/2} \sinh \sqrt{U})_{ij} u_j^A. \quad (2.4)$$

Under the transformation $g_o \Omega_o = \Omega'_o h_o$, the coset transforms as $\Omega_o \rightarrow \Omega'_o$, and the Maurer-Cartan one-forms transform according to

$$\Omega'^{-1}_o d\Omega'_o = h_o(\Omega_o^{-1} d\Omega_o) h_o^{-1} + h_o dh_o^{-1}. \quad (2.5)$$

Therefore, it can be concluded from Eq. (2.5) that the covariant coordinate differentials have the transformation property

$$\begin{aligned} i\omega'^A P_A &= id x'^a e'_{oa}{}^A P_A \\ &= e^{i(bM+\rho T)} id x^b e_{ob}^B P_B e^{-i(bM+\rho T)} = id x^b e_{ob}^B L_B^A P_A \end{aligned} \quad (2.6)$$

where L_B^A is the representation of the local unbroken H_o symmetry with vector indices, and H_o is spanned by the set of generators $\{M^{ab}, T\}$. As a result, the transformation of Zweibein induced by Eq. (2.6) has the form

$$e'_{oa}{}^A = \frac{\partial x^b}{\partial x'^a} e_{ob}^B L_B^A. \quad (2.7)$$

After eliminating the non-dynamic superfluous fields u_i^a with imposing the covariant constraint $\omega_z = 0$ in Eq. (2.2) by the inverse Higgs mechanism [29], the metric tensor of the two dimensional world-sheet is found to be

$$g_{ab}^o = e_{oa}^A e_{ob}^B \eta_{AB} = (\eta_{ab} - \partial_a \phi_i \partial_b \phi_i). \quad (2.8)$$

On the other hand, the formation of the vortex topological defect breaks the color-flavor locking symmetry $SU(N)_{C+F}$ to the stability subgroup $SU(N-1) \times U(1)$, and this embedding is rather a dynamical process. Therefore, the dynamics of the vortex-string corresponding to the internal orientational modes is described by these collective coordinates of the internal coset space $SU(N)/[SU(N-1) \times U(1)]$. Consider the Lie algebra of $SU(N)$ group, which has $N^2 - 1$ generators. In the fundamental representation they have the form

$$\begin{aligned} (T_{a'b'}^1)_{c'd'} &= (\delta_{a'c'} \delta_{b'd'} + \delta_{b'c'} \delta_{a'd'}), \\ (T_{a'b'}^2)_{c'd'} &= -i(\delta_{a'c'} \delta_{b'd'} - \delta_{b'c'} \delta_{a'd'}) \end{aligned} \quad (2.9)$$

where $a', b', c', d' = 1, 2, 3, \dots, N$ and $a' < b'$. The diagonal generators can be written as

$$(T_{a'}^3)_{c'd'} = \begin{cases} \delta_{c'd'} \sqrt{\frac{2}{a'(a'-1)}}, & c' < a' \\ -\delta_{c'd'} \sqrt{\frac{2(a'-1)}{a'}}, & c' = a' \\ 0, & c' > a' \end{cases} \quad (2.10)$$

where $2 \leq a' \leq N$. These matrices have the normalization condition

$$\text{Tr}(T_{A'} T_{B'}) = 2\delta_{A'B'} \quad (2.11)$$

where A', B' are from 1 to $N^2 - 1$. Likewise, the generators of $SU(N-1)$ group can be taken as

$$\begin{aligned} (T_{a'b'}^1)_{c'd'} &= (\delta_{a'c'}\delta_{b'd'} + \delta_{b'c'}\delta_{a'd'}), \\ (T_{a'b'}^2)_{c'd'} &= -i(\delta_{a'c'}\delta_{b'd'} - \delta_{b'c'}\delta_{a'd'}) \end{aligned} \quad (2.12)$$

with $a', b', c', d' = 1, 2, 3, \dots, N-1$ and $a' < b'$, and the generator of the $U(1)$ group has the diagonal traceless form

$$(T_N^3)_{c'd'} = \begin{pmatrix} \sqrt{2/[N(N-1)]} & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \sqrt{2/[N(N-1)]} & \\ & & & & & -\sqrt{2(N-1)/N} \end{pmatrix} \quad (2.13)$$

i.e.

$$(T_N^3)_{c'd'} = \begin{cases} \delta_{c'd'} \sqrt{\frac{2}{N(N-1)}}, & c' < N \\ -\delta_{c'd'} \sqrt{\frac{2(N-1)}{N}}, & c' = N \end{cases} \quad (2.14)$$

In addition, the broken generators associated with the coset $SU(N)/[SU(N-1) \times U(1)]$ are given by $N-1$ generators of the first type

$$(T_{a'N}^1)_{c'd'} = (\delta_{a'c'}\delta_{Nd'} + \delta_{Nc'}\delta_{a'd'}) \quad (2.15)$$

along with the other $N - 1$ generators of the second type

$$(T_{a'N}^2)_{c'd'} = -i(\delta_{a'c'}\delta_{Nd'} - \delta_{Nc'}\delta_{a'd'}) \quad (2.16)$$

in which $a' = 1, 2, 3, \dots, N - 1$, and $c', d' = 1, 2, 3, \dots, N$. Therefore we have total $N^2 - 1 - ((N - 1)^2 - 1) + 1 = 2N - 2$ generators corresponding to the coset space, which have explicit forms

$$\begin{aligned} T^{1a'} = T_{a'N}^1 &= \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & & 0 \\ 0 & & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & \ddots & 1 \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \\ T^{2a'} = T_{a'N}^2 &= \begin{pmatrix} 0 & \cdots & 0 & -i \\ \vdots & \ddots & & 0 \\ 0 & & \ddots & \vdots \\ i & 0 & \cdots & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & \ddots & -i \\ 0 & \cdots & i & 0 \end{pmatrix}. \end{aligned} \quad (2.17)$$

Therefore, the coset representative elements $\Omega_{\mathcal{I}} = G_{\mathcal{I}}/H_{\mathcal{I}} = SU(N)/[SU(N - 1) \times U(1)]$ with respect to the stability group $H_{\mathcal{I}}$ can be exponentially parameterized as

$$\Omega_{\mathcal{I}} = e^{i(\phi^{1a'} T^{1a'} + \phi^{2a'} T^{2a'})}. \quad (2.18)$$

The parameters $\phi^{1a'}$ and $\phi^{2a'}$ are the Nambu-Goldstone (NG) fields corresponding to the spontaneous breaking of the full $SU(N)_{C+F}$ group. They transform nonlinearly under the left action of the general group elements $g_{\mathcal{I}}$ on the coset representative itself, i.e. $g_{\mathcal{I}}\Omega_{\mathcal{I}} = \Omega'_{\mathcal{I}}h_{\mathcal{I}}$, and the resulting inhomogeneous terms of the NG transformations directly signal the breakdown of the $SU(N)_{C+F}$ symmetry. The effective action of the string that describes its orientational modes in the internal space can be derived by using the metric tensor on the coset manifold, which is constructed from the Maurer-Cartan one-forms $\Omega_{\mathcal{I}}^{-1}d\Omega_{\mathcal{I}}$. In which the coset elements can be rewritten as

$$\Omega_{\mathcal{I}} = e^{i(\phi^{1a'} T^{1a'} + \phi^{2a'} T^{2a'})} = e^{i(\Phi^{i'} T^{i'} + \Phi^{*i'} T^{i'\dagger})} \quad (2.19)$$

where $\Phi^{i'} = \phi^{1i'} + i\phi^{2i'}$, $\Phi^{*i'} = \phi^{1i'} - i\phi^{2i'}$ with ϕ and Φ real and complex scalar

fields, respectively, and

$$\begin{aligned}
T^{i'} &= \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & 0 \\ 0 & & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & 0 \\ 0 & & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & \ddots & 0 \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \\
T^{i'\dagger} &= \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & & 0 \\ 0 & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & 1 \\ 0 & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix},
\end{aligned} \tag{2.20}$$

where $i' = 1, 2, 3, \dots, N-1$. Accordingly, these matrices have explicit multiplication relations as follows

$$\begin{aligned}
T^{i'} T^{j'} &= 0; T^{i'\dagger} T^{j'\dagger} = 0; T^{i'} T^{j'\dagger} = T \delta^{i'j'}; \\
T^{i'\dagger} T^{j'} &= M^{i'j'}; T^{i'} T = 0; T^{j'\dagger} T = T^{j'\dagger}; \\
T T^{i'} &= T^{i'}; T T^{i'\dagger} = 0; T T = T; \\
T^{i'} M^{j'k'} &= T^{k'} \delta^{i'j'}; T^{i'\dagger} M^{j'k'} = 0; M^{i'j'} T^{k'} = 0; \\
M^{k'j'} T^{i'\dagger} &= T^{k'\dagger} \delta^{i'j'}; M^{i'j'} T = 0; T M^{i'j'} = 0;
\end{aligned} \tag{2.21}$$

where

$$T = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & 0 \\ 0 & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{2.22}$$

and

$$M^{11} = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad M^{12} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \dots \tag{2.23}$$

$$M^{12} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & \ddots & & 0 \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad M^{21} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \dots \quad (2.24)$$

After normalizing the Nambu-Goldstone fields in the coset representative element,

$$\Omega_{\mathcal{I}} = e^{if(|\Phi|)(\frac{\Phi^{i'}}{|\Phi|}T^{i'} + \frac{\Phi^{*i'}}{|\Phi|}T^{i'})} \quad (2.25)$$

in which $f(|\Phi|)$ is the normalization function determined below, and the norm is defined as $|\Phi| = \sqrt{\Phi^{i'}\Phi^{*i'}}$, we can explicitly expand it with respect to the $G_{\mathcal{I}}$ generators, i.e.

$$\begin{aligned} \Omega_{\mathcal{I}} = & 1 + (\cos(f(|\Phi|)) - 1)T + \frac{\cos(f(|\Phi|)) - 1}{|\Phi|^2} \Phi^{*i'} \Phi^{j'} M^{i'j'} \\ & + i \frac{\sin(f(|\Phi|))}{|\Phi|} (\Phi^{*i'} T^{i'\dagger} + \Phi^{i'} T^{i'}) \end{aligned} \quad (2.26)$$

likewise

$$\begin{aligned} \Omega_{\mathcal{I}}^{-1} = & 1 + (\cos(f(|\Phi|)) - 1)T + \frac{\cos(f(|\Phi|)) - 1}{|\Phi|^2} \Phi^{*i'} \Phi^{j'} M^{i'j'} \\ & - i \frac{\sin(f(|\Phi|))}{|\Phi|} (\Phi^{*i'} T^{i'\dagger} + \Phi^{i'} T^{i'}). \end{aligned} \quad (2.27)$$

The Maurer-Cartan one-forms is then found to be

$$\Omega_{\mathcal{I}}^{-1} d\Omega_{\mathcal{I}} = i(A^{a'} T^{a'} + B^{a'} T^{a'\dagger} + CT + D^{a'b'} M^{a'b'}) \quad (2.28)$$

where

$$\begin{aligned} A^{a'} &= \cos(f(|\Phi|)) d \left(\sin(f(|\Phi|)) \frac{\Phi^{a'}}{|\Phi|} \right) \\ &\quad - \frac{\sin(f(|\Phi|))}{|\Phi|} d \left[\{ \cos(f(|\Phi|)) - 1 \} \frac{\Phi^{*j'} \Phi^{a'}}{|\Phi|^2} \right] \Phi^{j'} \\ &= df \frac{\Phi^{a'}}{|\Phi|} - (\cos f - 1) \sin f \frac{\Phi^{a'}}{|\Phi|} d \left(\frac{\Phi^{*j'}}{|\Phi|} \right) \frac{\Phi^{j'}}{|\Phi|} + \sin f d \left(\frac{\Phi^{a'}}{|\Phi|} \right); \\ B^{a'} &= df \frac{\Phi^{*a'}}{|\Phi|} - (\cos f - 1) \sin f \frac{\Phi^{*a'}}{|\Phi|} d \left(\frac{\Phi^{j'}}{|\Phi|} \right) \frac{\Phi^{*j'}}{|\Phi|} + \sin f d \left(\frac{\Phi^{*a'}}{|\Phi|} \right); \\ C &= -i \sin f \frac{\Phi^{a'}}{|\Phi|} d \left(\sin f \frac{\Phi^{*a'}}{|\Phi|} \right) + \cos f \sin f df; \\ D^{a'b'} &= -i \sin f \frac{\Phi^{*a'}}{|\Phi|} d \left(\sin f \frac{\Phi^{b'}}{|\Phi|} \right) - id \left[(\cos f - 1) \frac{\Phi^{*a'} \Phi^{b'}}{|\Phi|^2} \right] \\ &\quad - i(\cos f - 1) \frac{\Phi^{*a'} \Phi^{c'}}{|\Phi|^2} d \left[(\cos f - 1) \frac{\Phi^{*c'} \Phi^{b'}}{|\Phi|^2} \right]. \end{aligned} \quad (2.29)$$

Under the action $g_{\mathcal{I}}\Omega_{\mathcal{I}} = \Omega'_{\mathcal{I}}h_{\mathcal{I}}$, one can find that the Maurer-Cartan one-forms transform according to

$$\Omega'^{-1}_{\mathcal{I}}d\Omega'_{\mathcal{I}} = h_{\mathcal{I}}(\Omega^{-1}_{\mathcal{I}}d\Omega_{\mathcal{I}})h^{-1}_{\mathcal{I}} + h_{\mathcal{I}}dh^{-1}_{\mathcal{I}}. \quad (2.30)$$

Among them, the one-forms associated with the broken generators $T^{a'}$ and $T^{a'\dagger}$ transform covariantly, and the one-forms associated with the unbroken generators T and $M^{a'b'}$ transform inhomogeneously.

In addition, after rewriting the one-forms $A^{a'}$ as

$$\begin{aligned} A^{a'} = & \frac{df}{d|\Phi|} \frac{\Phi^{a'}\Phi^{*i'}d\Phi^{i'}}{2|\Phi|^2} + (\cos f - 1) \sin f \frac{\Phi^{a'}}{2|\Phi|^2} \frac{\Phi^{*i'}d\Phi^{i'}}{|\Phi|} \\ & + \sin f \left(-\frac{\Phi^{*i'}d\Phi^{i'}\Phi^{a'}}{2|\Phi|^3} + \frac{d\Phi^{a'}}{|\Phi|} \right) + \frac{df}{d|\Phi|} \frac{\Phi^{a'}d\Phi^{*i'}\Phi^{i'}}{2|\Phi|^2} \\ & - (\cos f - 1) \sin f \frac{\Phi^{a'}}{2|\Phi|^2} \frac{\Phi^{i'}d\Phi^{*i'}}{|\Phi|} - \sin f \frac{d\Phi^{*i'}\Phi^{i'}\Phi^{a'}}{2|\Phi|^3} \end{aligned} \quad (2.31)$$

one can obviously note that the two covariant parts, which correspond to $d\Phi$ and $d\Phi^*$ terms respectively, transform independently under Eq. (2.30). Imposing the covariant condition on this one-forms by setting the term related to $d\Phi^*$ to zero gives us

$$\left(\frac{df}{d|\Phi|} \frac{1}{2|\Phi|^2} - \cos f \sin f \frac{1}{2|\Phi|^3} \right) \Phi^{a'}d\Phi^{*i'}\Phi^{i'} = 0. \quad (2.32)$$

Then the normalization function can be secured as

$$df/d|\Phi| = \cos f \sin f \frac{1}{|\Phi|}, \quad (2.33)$$

i.e.

$$f = \arctan(|\Phi|). \quad (2.34)$$

Therefore, $A^{a'}$ becomes

$$\begin{aligned}
A^{a'} &= \frac{d\Phi^{i'} \Phi^{*i'}}{1 + |\Phi|^2} \frac{\Phi^{a'}}{2|\Phi|^2} + \sin f \frac{1}{|\Phi|} d\Phi^{a'} - \sin f \Phi^{*i'} \frac{d\Phi^{i'}}{|\Phi|^3} \Phi^{a'} \\
&\quad + \cos f \sin f \Phi^{*i'} \frac{d\Phi^{i'}}{2|\Phi|^3} \Phi^{a'} \\
&= \frac{1}{\sqrt{1 + |\Phi|^2}} \left(d\Phi^{a'} - \frac{1}{|\Phi|^2} \Phi^{*i'} d\Phi^{i'} \Phi^{a'} + \frac{d\Phi^{i'} \Phi^{*i'}}{\sqrt{1 + |\Phi|^2}} \frac{\Phi^{a'}}{2|\Phi|^2} \right. \\
&\quad \left. + \frac{1}{\sqrt{1 + |\Phi|^2}} \Phi^{*i'} \frac{d\Phi^{i'}}{2|\Phi|^2} \Phi^{a'} \right) \\
&= \frac{1}{\sqrt{1 + |\Phi|^2}} \left[d\Phi^{a'} + \left(-\frac{1}{|\Phi|^2} + \frac{1}{\sqrt{1 + |\Phi|^2}} \frac{1}{|\Phi|^2} \right) \Phi^{*i'} \Phi^{a'} d\Phi^{i'} \right] \quad (2.35)
\end{aligned}$$

where $A^{a'}$ is the covariant coordinate differential, and $d\Phi^{i'}$ is the general coordinate differential on the coset manifold. The vielbein can be derived accordingly

$$\begin{aligned}
A^{a'} &= d\Phi^{i'} e_{\mathcal{I}i'}^{a'} \\
&= d\Phi^{i'} \frac{1}{\sqrt{1 + |\Phi|^2}} \left[\delta_{i'}^{a'} + \left(-\frac{1}{|\Phi|^2} + \frac{1}{\sqrt{1 + |\Phi|^2}} \frac{1}{|\Phi|^2} \right) \Phi^{*i'} \Phi^{a'} \right]. \quad (2.36)
\end{aligned}$$

Thus the vielbein becomes

$$e_{\mathcal{I}i'}^{a'} = \frac{1}{\sqrt{1 + |\Phi|^2}} \left[\delta_{i'}^{a'} + \left(-\frac{1}{|\Phi|^2} + \frac{1}{\sqrt{1 + |\Phi|^2}} \frac{1}{|\Phi|^2} \right) \Phi^{*i'} \Phi^{a'} \right]. \quad (2.37)$$

The $SU(N-1) \times U(1)$ invariant interval is therefore given by

$$\begin{aligned}
ds_{\mathcal{I}}^2 &= A^{a'} A^{*a'} = e_{\mathcal{I}i'}^{a'} e_{\mathcal{I}j'}^{*b'} d\Phi^{i'} d\Phi^{*j'} \delta_{b'}^{a'} \\
&= g_{ij}^{\mathcal{I}} d\Phi^i d\Phi^{*j} \quad (2.38)
\end{aligned}$$

where the metric $g_{ij}^{\mathcal{I}}$ is

$$g_{ij}^{\mathcal{I}} = e_{\mathcal{I}i'}^{a'} e_{\mathcal{I}j'}^{*a'}. \quad (2.39)$$

Considering Eq. (2.37), the invariant interval of the internal space becomes

$$ds_{\mathcal{I}}^2 = g_{ij}^{\mathcal{I}} d\Phi^i d\Phi^{*j} = \frac{1}{1 + |\Phi|^2} \left[d\Phi^{i'} d\Phi^{*i'} - \frac{1}{1 + |\Phi|^2} \Phi^{j'} \Phi^{*i'} d\Phi^{i'} d\Phi^{*j'} \right], \quad (2.40)$$

where the metric $g_{ij}^{\mathcal{I}}$ can be obtained, to yield

$$g_{ij}^{\mathcal{I}} = \frac{1}{1 + |\Phi|^2} \left[\delta_{ij} + \frac{1}{|\Phi|^2} \left(\frac{1}{1 + |\Phi|^2} - 1 \right) \Phi^{j'} \Phi^{*i'} \right]. \quad (2.41)$$

We thus have obtained the Fubini-Study metric on the complex projective space \mathbf{CP}^{N-1} as expected.

Now we are ready to construct the low energy effective action of the vortex-string. Under the general coordinate transformations of Eq. (2.7), by using Eq. (2.8) one can find the following invariance

$$\begin{aligned} d^2x' \sqrt{|\det g^{o'}|} &= d^2x' \det e'^A_{ob} = d^2x \det \left| \frac{\partial x'^b}{\partial x^a} \right| \det \left| \frac{\partial x^b}{\partial x'^a} \right| \det e^B_{ob} \det L^A_B \\ &= d^2x \det e^B_{ob} = d^2x \sqrt{|\det g^o|} \end{aligned} \quad (2.42)$$

along with the invariant internal space interval

$$ds_{\mathcal{I}}^2 = g^{\mathcal{I}}_{i'j'*} d\Phi^{i'} d\Phi^{*j'} = ds'^2_{\mathcal{I}} = g^{\mathcal{I}'}_{i'j'*} d\Phi'^{i'} d\Phi'^{*j'}. \quad (2.43)$$

Hence,

$$d^2x \sqrt{|\det g^o|} g^{\mathcal{I}}_{i'j'*} \partial_a \Phi^{i'} \partial^a \Phi^{*j'} = d^2x' \sqrt{|\det g^{o'}|} g^{\mathcal{I}'}_{i'j'*} \partial'_a \Phi'^{i'} \partial'^a \Phi'^{*j'} \quad (2.44)$$

is invariant under both the spacetime and internal space transformations.

Considering Eq. (2.8) the action of the string described by the Nambu-Goldstone oscillation modes corresponding to its low energy oscillations into the covolume space takes the form

$$\begin{aligned} S_o &= -T \int d^2x \sqrt{|\det g^o|} = -T \int d^2x \sqrt{|\det(\eta_{ab} - \partial_a \phi_i \partial_b \phi_i)|} \\ &= -T \int d^2x \sqrt{\det(\delta_{ij} - \partial_a \phi_i \partial^a \phi_j)} \end{aligned} \quad (2.45)$$

where T stands for string tension, and it is obviously $SO(1,1) \times SO(2)$ invariant. This part is the Nambu-Goto action [30] which is sufficient for describing the dynamics of an Abelian vortex-string.

In addition to the Nambu-Goto action (2.45) for the broken translational zero mode, there exist the Nambu-Goldstone modes for the internal symmetry breaking in the case of non-Abelian vortex-strings. The effective action that describes the dynamics of the string in both the spacetime and internal space breakings is then

given by

$$\begin{aligned}
S_{\mathcal{I}} &= T_{\mathcal{I}} \int d^2x \sqrt{|\det g^o| g_{i'j'}^{\mathcal{I}} \partial_a \Phi^{i'} \partial^a \Phi^{*j'}} \\
&= T_{\mathcal{I}} \int d^2x \sqrt{\det(\delta_{ij} - \partial_a \phi_i \partial^a \phi_j)} \\
&\quad \times \frac{1}{1 + |\Phi|^2} \left[\partial_a \Phi^{i'} \partial^a \Phi^{*i'} - \frac{1}{1 + |\Phi|^2} \Phi^{j'} \Phi^{*i'} \partial_a \Phi^{i'} \partial^a \Phi^{*j'} \right] \\
&= T_{\mathcal{I}} \int d^2x \left\{ \partial_a \Phi^{i'} \partial^a \Phi^{*i'} - \frac{1}{2} \partial_a \phi^i \partial^a \phi^i \partial_b \Phi^{i'} \partial^b \Phi^{*i'} \right. \\
&\quad \left. + \mathcal{O}(\Phi \Phi^* \partial \Phi \partial \Phi^*) + \mathcal{O}(\partial \Phi \partial \Phi^* (\partial \phi)^4) \right\}, \tag{2.46}
\end{aligned}$$

where $T_{\mathcal{I}}$ is the coupling between the space-time and the internal zero modes. Therefore one can find that the effective action has a synthesis form of the Nambu-Goto- \mathbf{CP}^{N-1} model actions $S_o + S_{\mathcal{I}}$. Note that the moduli space is the direct product $\mathbf{C} \times \mathbf{CP}^{N-1}$ of the translational and the internal zero modes and consequently the action is decomposed into them at the order of the quadratic derivatives, but there exist the interaction terms at the quartic (and higher) derivatives.

Typically, the coherence length and penetration depth of the vortex are characterized by the Compton wave lengths of massive Higgs and gauge bosons respectively. Here, we have assumed that the width or transverse size of a vortex-string is much smaller than the energy scale which we consider. On the other hand, one may consider the correction to the action by taking account the effect of width of the vortex. In order to do so, it is necessary to add the effective action terms that characterizes this embedding of the world-sheet in the target spacetime [27, 28]. The interactions that describes the stiffness or rigidity of the string can be shown as

$$S' = k_0 \int d^2x \sqrt{|\det(\eta_{ab} - \partial_a \phi^i \partial_b \phi^i)|} K_{ab}^i K^{iab} \tag{2.47}$$

in which the coupling constant k_0 stands for the rigidity or stiffness, and the extrinsic curvatures K_{ab}^i has the form $K_{ab}^i = n_{\mu}^i \partial_a \partial_b x^{\mu}$ with the normalization condition $n_{\mu}^i n^{j\mu} = \delta^{ij}$ for the unit norm vectors n_{μ}^i . As a result, these extrinsic curvatures terms produce interactions containing quartic derivatives in the strings actions. In general for a p -brane the intrinsic scalar curvature R on its world-volume can be written as $R = (K_a^{ia})^2 - K_b^{ia} K_a^{ib}$. In the case of the string, this is a total derivative term because of the Euler's theorem, and so we do not need it [27]. Up to the same

order corrections, the string actions can also be supplemented with the following but not scale invariant interactions

$$S_1 = k_1 \int d^2x \sqrt{|\det g^o|} (\partial_a x^\mu \partial^a x_\mu)^2 + k_2 \int d^2x \sqrt{|\det g^o|} \partial_a x^\mu \partial^a x_\nu \partial_b x_\mu \partial^b x^\nu. \quad (2.48)$$

Likewise, we have the following quartic coupling terms for the internal moduli

$$\begin{aligned} S_2 = & k'_1 \int d^2x \sqrt{|\det g^o|} (g_{i'j'}^\mathcal{I} \partial_a \Phi^{i'} \partial^a \Phi^{*j'})^2 \\ & + k'_2 \int d^2x \sqrt{|\det g^o|} g_{i'j'}^\mathcal{I} \partial_a \Phi^{i'} \partial^b \Phi^{*j'} g_{k'l'}^\mathcal{I} \partial_b \Phi^{k'} \partial^a \Phi^{*l'} \\ & + k'_3 \int d^2x \sqrt{|\det g^o|} g_{i'j'}^\mathcal{I} \partial_a \Phi^{i'} \partial^b \Phi^{*j'} g_{k'l'}^\mathcal{I} \partial_b \Phi^{*l'} \partial^a \Phi^{k'} \end{aligned} \quad (2.49)$$

where the metric $g_{i'j'}^\mathcal{I}$ is given in Eq. (2.41).¹ Therefore, considering Eqs. (2.45),

¹ In more general, the following generally covariant terms are also invariant under the holomorphic isometry $G_\mathcal{I}$ for the Kähler internal target spaces: $\int d^2x \sqrt{|\det g^o|} R^\mathcal{I}$, $\int d^2x \sqrt{|\det g^o|} R^\mathcal{I} g_{i'j'}^\mathcal{I} \partial_a \Phi^{i'} \partial^a \Phi^{*j'}$, $\int d^2x \sqrt{|\det g^o|} R_{i'j'}^\mathcal{I} \partial_a \Phi^{i'} \partial^a \Phi^{*j'}$, $\int d^2x \sqrt{|\det g^o|} R_{i'j'k'l'}^\mathcal{I} \partial_a \Phi^{i'} \partial^a \Phi^{*j'} \partial_b \Phi^{k'} \partial^b \Phi^{*l'}$ and so on, where $R^\mathcal{I}$, $R_{i'j'}^\mathcal{I}$ and $R_{i'j'k'l'}^\mathcal{I}$ are the scalar curvature, the Ricci-form and the Riemann curvature tensor of the target Kähler manifold, respectively. This is because (holomorphic) isometries are subgroups of the (holomorphic) general coordinate transformation. However those terms are not independent from the terms in Eq. (2.49) in the case of \mathbf{CP}^{N-1} which we are concerned with because of the identities $R_\mathcal{I} = \text{const.}$, $g_{i'j'}^\mathcal{I} = N R_{i'j'}^\mathcal{I}$ (Einstein), and $R_{i'j'k'l'}^\mathcal{I} \sim (g_{i'j'}^\mathcal{I} g_{k'l'}^\mathcal{I} + g_{i'l'}^\mathcal{I} g_{k'j'}^\mathcal{I})$ (symmetric space), see e.g. [31]. The so-called Kähler normal coordinates should be useful for the expansion of geometric quantities in the internal Kähler manifold [32]. The above terms are needed in general for arbitrary Kähler target spaces.

(2.46), (2.47), (2.48) and (2.49), the effective actions of the string amount to

$$\begin{aligned}
S = & -T \int d^2x \sqrt{\det(\delta_{ij} - \partial_a \phi_i \partial^a \phi_j)} \\
& + T_{\mathcal{I}} \int d^2x \sqrt{|\det g^o|} \frac{1}{1 + |\Phi|^2} \left[\partial_a \Phi^{i'} \partial^a \Phi^{*i'} - \frac{1}{1 + |\Phi|^2} \Phi^{j'} \Phi^{*i'} \partial_a \Phi^{i'} \partial^a \Phi^{*j'} \right] \\
& + k_1 \int d^2x \sqrt{|\det g^o|} K_{ab}^i K^{iab} \\
& + k_2 \int d^2x \sqrt{|\det g^o|} (\partial_a x^\mu \partial^a x_\mu)^2 \\
& + k_3 \int d^2x \sqrt{|\det g^o|} \partial_a x^\mu \partial^a x_\nu \partial_b x_\mu \partial^b x^\nu \\
& + k'_1 \int d^2x \sqrt{|\det g^o|} g_{i'j'^*}^{\mathcal{I}} \partial_a \Phi^{i'} \partial^a \Phi^{*j'} g_{k'l'^*}^{\mathcal{I}} \partial_b \Phi^{k'} \partial^b \Phi^{*l'} \\
& + k'_2 \int d^2x \sqrt{|\det g^o|} g_{i'j'^*}^{\mathcal{I}} \partial_a \Phi^{i'} \partial^b \Phi^{*j'} g_{k'l'^*}^{\mathcal{I}} \partial_b \Phi^{k'} \partial^a \Phi^{*l'} \\
& + k'_3 \int d^2x \sqrt{|\det g^o|} g_{i'j'^*}^{\mathcal{I}} \partial_a \Phi^{i'} \partial^b \Phi^{*j'} g_{k'l'^*}^{\mathcal{I}} \partial_b \Phi^{*l'} \partial^a \Phi^{k'}. \tag{2.50}
\end{aligned}$$

As a summary, the first term corresponds to the vortex low energy fluctuation in the target spacetime; the second term describes the interactions between the internal orientational moduli and the spatial moduli, which result quartic derivative coupling between these two modes. The third term is the extrinsic curvature coupling as the characteristics of the embedding of the string in the higher dimensions. The last five terms supplement us with other quartic derivative couplings with respect to spacetime and internal space modes respectively, and $k_i, k'_{i'}$ are coupling constants. Among the coupling constants, T, k_i ($i = 1, 2, 3$) exist in the Abelian vortex-string of $N = 1$ while $T_{\mathcal{I}}, k'_{i'}$ ($i' = 1, 2, 3$) are peculiar to the non-Abelian vortex-string for $N > 1$.

Here we make a comment on the microscopic derivation of those effective coupling constants, $T, T_{\mathcal{I}}, k_i, k'_{i'}$ in the effective Lagrangian (2.50). In principle we can determine those coupling constants from a given microscopic Lagrangian. Let us consider a Bogomol'nyi-Prasad-Sommerfield (BPS) vortex-string in the non-Abelian $U(N)$ gauge theory coupled to N Higgs scalar fields in the fundamental representation with the common $U(1)$ charges. First, the tension T can be calculated to be $T = 2\pi v^2$ with v the VEV of the Higgs fields [5, 6]. Second, the coupling between space-time and internal modes, $T_{\mathcal{I}}$, called the Kähler class, can be determined to be $T_{\mathcal{I}} = 4\pi/g^2$ with the $U(N)$ gauge coupling constant g [7, 8, 9, 10]. In the case

of a non-Abelian vortex-string in dense QCD [13, 14], the tension T is given by $T = 2\pi v^2 \log \Lambda$ with an infrared cutoff Λ . The coupling $T_{\mathcal{I}}$ has been calculated recently in [15] as $T_{\mathcal{I}} = 4\pi c/g^2$ with some numerical constant c which is not in general unity but depends on the parameters in the original Lagrangian.

On the other hand, higher order terms are in general difficult to be determined. There have been many attempts to calculate k_i ($i = 1, 2, 3$) for the Abelian ($N = 1$) local vortex [33], but it seems that many discussions on rigidity or stiffness have been given without agreement with each other. For the non-Abelian $U(N)$ case, there are three more parameters $k'_{i'}$ ($i' = 1, 2, 3$) at this order, which have not been calculated yet. Our work will provide a general basis for a microscopic derivation of the higher order effective action.

Recently non-Abelian vortices have been extended from the $U(N)$ gauge group to gauge groups $U(1) \times G$ where G is $SO(N)$, $USp(N)$ [34] or further arbitrary Lie groups [35]. Accordingly, the orientational zero modes become $SO(2N)/U(N)$, $USp(2N)/U(N)$ and so on. An extension to those cases is straightforward since the construction of those coset spaces is known [36].

3 General p -Brane Dynamics

In the previous section, the effective actions of the vortex-string ($p = 1$ brane) that describe both the spacetime and internal space spontaneous symmetry breakings have been discussed. To generalize, we consider a general p -brane topological defect whose formation not only breaks the embedded target spacetime symmetry but also spontaneously breaks the internal target group $G_{\mathcal{I}}$ down to an invariant subgroup $H_{\mathcal{I}}$. Therefore, the dynamics of the p -brane would be described by the associated Nambu-Goldstone modes corresponding to the collective degrees of freedom of the enlarged coset space. For instance, domain walls ($p = 2$ brane in $D = 3 + 1$) have $U(N)$ orientational moduli in a $U(N)$ gauge theory with $2N$ Higgs scalar fields with properly degenerated masses [37].² Non-Abelian monopoles, Skyrmons ($p = 0$ branes in $D = 3 + 1$) and Yang-Mills instantons ($p = -1$ brane in $D = 3 + 1$) also

² When the Higgs masses are non-degenerated, each domain wall has only $U(1)$ internal modulus in addition to translational zero mode [38].

have orientational moduli in general.

Consider such a p -brane embedded in the target D dimensional flat spacetime. As a result, its moduli space is given by

$$ISO(1, D-1)/[SO(1, p) \times SO(D-p-1)] \otimes G_{\mathcal{I}}/H_{\mathcal{I}} \quad (3.1)$$

where the spacetime stability group of the p -brane takes the form $SO(1, p) \times SO(D-p-1)$, namely, the brane has Lorentz invariant $SO(1, p)$ symmetry in $p+1$ dimensional spacetime along with the rotation invariance in the $D-p-1$ codimensions, and its long wave fluctuations are described by the collective coordinates parameterized by the coset space Ω_o (see Eq. (3.2)). On the other hand, the internal moduli space is given by $\Omega_{\mathcal{I}} = G_{\mathcal{I}}/H_{\mathcal{I}}$, illustrating that the orientation of the brane in the internal space breaks the symmetry group $G_{\mathcal{I}}$ down to $H_{\mathcal{I}}$, where $G_{\mathcal{I}}$ stands for a general compact, connected, semi-simple Lie group {formed by unbroken generators T^J and broken generators S^I }, and $H_{\mathcal{I}}$ is the unbroken subgroup {formed by generators T^J }. As a result, the coset representative elements are parameterized as

$$\Omega = \Omega_o \Omega_{\mathcal{I}} = e^{ix^\mu P_\mu} e^{i(\phi^m Z_m + u_\mu^m K_m^\mu)} e^{i\Phi^I S^I} \quad (3.2)$$

where x^μ are the coordinates that parameterize the p -brane world volume in the static gauge, with $\mu = 0, 1, \dots, p$; Z_m are the broken generators associated with the translational directions transverse to the brane, with $m = 1, \dots, D-p-1$; K_m^μ are the broken generators related to the rotations that mix the brane world volume and the codimensional directions. In the sector of the internal space, the generators T^J and S^I have the following commutation relations:

$$[T, T] \propto T, [T, S] \propto S \quad (3.3)$$

i.e., the generators S^I form a representation for the subgroup $H_{\mathcal{I}}$. Accordingly, the complete Maurer-Cartan one-forms can be written as

$$\begin{aligned} \Omega^{-1} d\Omega &= (\Omega_o \Omega_{\mathcal{I}})^{-1} d(\Omega_o \Omega_{\mathcal{I}}) \\ &= \Omega_o^{-1} d\Omega_o + \Omega_{\mathcal{I}}^{-1} d\Omega_{\mathcal{I}} \\ &= i(\omega^{A'} P_{A'} + \omega_{zmi} Z_m + \omega_{A'}^m K_m^{A'} \dots + A^M S^M + B^N T^N) \end{aligned} \quad (3.4)$$

with

$$\begin{aligned}
i\omega^{A'} P_{A'} &= id x^\mu P_{A'} (\delta_\mu^{A'} + u_{m\mu} (U^{-\frac{1}{2}} (\cosh \sqrt{4U} - 1) U^{-\frac{1}{2}}))_{mn} u_n^{A'} \\
&\quad - id \phi_m (U^{-\frac{1}{2}} \sinh \sqrt{4U})_{mn} u_n^{A'} P_{A'} = id x^\mu e_\mu^{A'} P_{A'}; \\
i\omega_{z_m} Z_m &= id \phi_n (\cosh \sqrt{4U})_{nm} Z_m - id x^\mu u_{n\mu} (\cosh \sqrt{4U} U^{-\frac{1}{2}} \tanh \sqrt{4U})_{nm} Z_m; \\
i\omega_{A'}^m K_m^{A'} &= id u_n^{B'} (\sinh \sqrt{M} M^{-1/2})_{nm B' A'} K_m^{A'}; \\
iA^M S^M &= id \Phi^I E_I^M S^M
\end{aligned} \tag{3.5}$$

in which $U_{mn} = u_m^\mu u_{n\mu}$, and $M_{mn\mu\nu} = 4(u_{m\mu} u_{n\nu} - 2u_{m\mu} u_{n\nu} + u_{l\mu} u_{l\nu} \delta_{mn})$. After eliminating the non-dynamical fields u_μ^m setting $\omega_{z_m} = 0$ by the inverse Higgs mechanism [29], the p -brane world volume metric is given by [25]

$$g_{\mu\nu}^o = e_{o\mu}^{A'} e_{o\nu}^{B'} \eta_{A'B'} = (\eta_{\mu\nu} - \partial_\mu \phi_m \partial_\nu \phi_m) \tag{3.6}$$

On the other hand, the metric of the internal coset manifold has the form

$$g_{JO}^{\mathcal{I}} = E_J^N E_O^M L_{NM} \tag{3.7}$$

where L_{NM} is the metric imposed on the covariant coordinate differentials for the invariant interval under the $H_{\mathcal{I}}$ transformation

$$ds_{\mathcal{I}}^2 = A^M A^N L_{MN} = g_{JO}^{\mathcal{I}} d\Phi^J d\Phi^O \tag{3.8}$$

Therefore, considering Eqs. (3.6) and (3.7), the metrics in the enlarged moduli space give us the effective theory of these moduli (NG) fields on the brane world volume.

The effective actions of p -brane dynamics can be written as³

$$\begin{aligned}
S = & -T \int d^{p+1}x \sqrt{|\det(\eta_{\mu\nu} - \partial_\mu \phi_m \partial_\nu \phi_m)|} + T_{\mathcal{I}} \int d^{p+1}x \sqrt{|\det g^o|} g_{JO}^{\mathcal{I}} \partial_\mu \Phi^J \partial^\mu \Phi^O \\
& + k_0 \int d^{p+1}x \sqrt{|\det g^o|} K^{m\mu\nu} K_{\mu\nu}^m + k_3 \int d^{p+1}x \sqrt{|\det g^o|} R \\
& + k_1 \int d^{p+1}x \sqrt{|\det g^o|} (\partial_\mu x^{\mu'} \partial^\mu x_{\mu'})^2 \\
& + k_2 \int d^{p+1}x \sqrt{|\det g^o|} \partial_\mu x^{\mu'} \partial^\mu x_{\nu'} \partial_\nu x_{\mu'} \partial^\nu x^{\nu'} \\
& + k'_1 \int d^{p+1}x \sqrt{|\det g^o|} g_{JO}^{\mathcal{I}} \partial_\mu \Phi^J \partial^\mu \Phi^O g_{KL}^{\mathcal{I}} \partial_\nu \Phi^K \partial^\nu \Phi^L \\
& + k'_2 \int d^{p+1}x \sqrt{|\det g^o|} g_{JO}^{\mathcal{I}} \partial_\mu \Phi^J \partial^\nu \Phi^O g_{KL}^{\mathcal{I}} \partial_\nu \Phi^K \partial^\mu \Phi^L \\
& + k'_3 \int d^{p+1}x \sqrt{|\det g^o|} R_{\mathcal{I}} + \dots
\end{aligned} \tag{3.9}$$

in which $x^{\mu'}$ are coordinates that parameterize the target spacetime, and $\mu' = 0, 1, 2, \dots, D-1$. The second term gives us the coupling modes between the internal and the spacetime symmetry breakings. As a result, the general p -brane effective actions in terms of both the spacetime and the internal $G_{\mathcal{I}}/H_{\mathcal{I}}$ space breakings have the synthesis form of Nambu-Goto-NLS model actions. The third term is the extrinsic curvature part, describing the rigidity or stiffness of the p -brane and specifying its motion in the target space, where $K_{\mu\nu}^m$ are the extrinsic curvatures, given by $K_{\mu\nu}^m = e_\mu^{B'} e_\nu^{A'} K_{B'A'}^m$, and $\omega_{A'}^m = dx^\mu e_\mu^{B'} K_{B'A'}^m$ [39]. In the brane-world scenario, the quartic derivative couplings are very crucial in brane dynamics, signaling the brane stiffness and embedding in the extra dimensions. Their phenomenological consequences have been explored in collider physics and cosmology as well [40]. The other terms are higher order corrections to the brane actions up to the order of quartic derivatives. Besides, additional fermionic degrees of freedom can also be located on the brane world volume to describe its oscillations into the superspace for BPS [21, 22] and non-BPS branes [23, 24] in supersymmetric theories.

Recently, in Ref. [41], a non-Abelian $p = 3$ brane topological defect in $D = 5 + 1$

³When the internal target space is Kähler, there also exist the terms in footnote 3 in general unless the curvature tensors are written by the metric tensor as in \mathbf{CP}^{N-1} . This is because the complex structure I_j^i of Kähler manifolds can be used to construct invariant terms in the effective action. When the target space has further invariant tensors, they can be also used to construct further invariant terms in the action.

with orientational moduli configurations has also been discussed. In brane world scenarios, standard model particles should be realized as low energy fluctuations localized on the $p = 3$ brane world volume. Phenomenologically, the physical consequences of the couplings in Eq. (3.9) for the $p = 3$ brane, which produce interactions containing quartic derivatives in addition to the extrinsic curvature terms, are open appealing problems to be explored.

4 Conclusion and Discussion

The non-Abelian vortex-strings were found in supersymmetric $U(N)$ gauge theories and high density QCD. They have the orientation zero modes in the internal color and flavor space. In this paper, we have constructed the low energy effective action for a non-Abelian vortex-string, the presence of which breaks the translational and the internal $SU(N)$ symmetries. It has been turned out to be the invariant synthesis of the Nambu-Goto- \mathbf{CP}^{N-1} model action in the lowest order. We then have constructed higher order terms related to the string rigidity in both the space-time and the internal spaces. Our work will provide a general basis to go on the microscopic derivation of the effective action of the non-Abelian vortex-string. We have also constructed the effective action of a p -brane which breaks the translational and the internal $G_{\mathcal{I}}$ symmetries at the same time. In appendix A we give a relation between the Nambu-Goto- \mathbf{CP}^1 model and the Skyrme-Faddeev model.

As demonstrated in this paper the nonlinear realization method offers a powerful tool to construct the low energy effective action of a single vortex. On the other hand multiple vortices admit more ample moduli spaces [11] but symmetry is not enough to determine the effective Lagrangian. However the moduli space can be written as a symmetric product $(\mathbf{C} \times \mathbf{CP}^{N-1})^k / S_k$ of each vortex moduli space when all vortices are well separated. So the nonlinear realization may work to some extent in this case because the position and the orientation of each vortex are approximate Nambu-Goldstone modes though exact Nambu-Goldstone modes are only overall translation and orientation.

We have studied only bosonic action in this paper. In $\mathcal{N} = 2$ supersymmetric gauge theories, $U(N)$ non-Abelian vortices are BPS states preserving a half of super-

symmetry [5, 6]. Therefore Nambu-Goldstone fermions associated with the partially broken supersymmetry are localized in the vortex-string. The effective theory is expected to become an invariant synthesis of the Nambu-Goto action with partial supersymmetry breaking terms [21] and the $\mathcal{N} = (2, 2)$ supersymmetric \mathbf{CP}^{N-1} model. Higher order terms should be severely restricted by supersymmetry. On the other hand, non-Abelian vortices can become non-BPS states in $\mathcal{N} = 1$ supersymmetric theories with an F-term potential. In that case, the effective action should be an invariant synthesis of the Nambu-Goto-Volkov-Akulov action [23] and the bosonic \mathbf{CP}^{N-1} model. Extensions of our work to those cases remain as a future problem.

In this paper we have focused on the case of local vortices with the number of flavor equal to the number of color, $N_F = N_C$. When the number of flavor is larger than the number of color, $N_F > N_C$, vortices have a size modulus and are called semi-local vortices [42]. In this case, the normalizability of zero modes does not hold automatically; the \mathbf{CP}^{N-1} modes are (non-)normalizable when size modulus is (non-)zero [43]. When zero modes are non-normalizable they do not appear in the effective theory on the vortex but rather should be interpreted as bulk modes. Therefore we have to pay attention not only to a symmetry structure but also to the normalizability of zero modes.

Acknowledgments

One of the authors (L.X.Liu) would like to thank T.K.Kuo and the THEP group in Physics Department at Purdue University for support. M.N. would like to thank T. E. Clark, S. T. Love and the THEP group in Department of Physics at Purdue University for warm hospitality during his stay where this work was initiated. The authors also would like to thank Martin Kruczenski for useful comments. The work of M.N. is supported in part by Grant-in-Aid for Scientific Research (No. 20740141) from the Ministry of Education, Culture, Sports, Science and Technology-Japan.

A The CP^1 Model with Four Derivative Terms

The Nambu-Goto- CP^N Lagrangian (2.50) reduces in the case of $N = 1$ with $p = 3$ to

$$\begin{aligned}
S = & -T \int d^4x \sqrt{\det(\delta_{ij} - \partial_a \phi_i \partial^a \phi_j)} \\
& + T_{\mathcal{I}} \int d^4x \sqrt{|\det g^o|} \frac{\partial_a \Phi \partial^a \Phi^*}{(1 + |\Phi|^2)^2} \\
& + k_0 \int d^4x \sqrt{|\det g^o|} K_{ab}^i K^{iab} \\
& + k_1 \int d^4x \sqrt{|\det g^o|} (\partial_a x^\mu \partial^a x_\mu)^2 + k_2 \int d^4x \sqrt{|\det g^o|} \partial_a x^\mu \partial^a x_\nu \partial_b x_\mu \partial^b x^\nu \\
& + k'_1 \int d^4x \sqrt{|\det g^o|} \frac{\partial_a \Phi \partial^a \Phi^* \partial_b \Phi \partial^b \Phi^*}{(1 + |\Phi|^2)^4} \\
& + k'_3 \int d^4x \sqrt{|\det g^o|} \frac{\partial_a \Phi \partial^a \Phi \partial_b \Phi^* \partial^b \Phi^*}{(1 + |\Phi|^2)^4}. \tag{A.1}
\end{aligned}$$

In this case we have noticed that the term with k'_2 is degenerate with k'_1 term.

Here we would like to compare this Lagrangian with the Skyrme-Faddeev model [44], which allows knot like solitons in 3+1 dimensions. Let us introduce the auxiliary gauge field and its field strength as the following

$$A_a = \frac{-i(\Phi^* \partial_a \Phi - \partial_a \Phi^* \cdot \Phi)}{2(1 + |\Phi|^2)} \tag{A.2}$$

$$F_{ab} = \partial_a A_b - \partial_b A_a = \frac{i(\partial_a \Phi^* \partial_b \Phi - \partial_b \Phi^* \partial_a \Phi)}{(1 + |\Phi|^2)^2}. \tag{A.3}$$

Then the so-called the Skyrme-Faddeev term [44] can be written as the field strength squared

$$F_{ab}^2 = 2 \frac{(\partial_a \Phi^* \partial^a \Phi)^2 - |\partial_a \Phi \partial^a \Phi|^2}{(1 + |\Phi|^2)^4}. \tag{A.4}$$

This term is also called the baby Skyrme term in $d = 2 + 1$ [45]. There is the other independent fourth order term [46, 47, 48] which is simply the kinetic term squared:

$$\frac{(\partial_a \Phi \partial^a \Phi^*)^2}{(1 + |\Phi|^2)^4}. \tag{A.5}$$

This term arises when one constructs the low energy effective theory of pure $SU(2)$ Yang-Mills theory [46]. It is also needed for supersymmetric generalization of the

Skyrme-Faddeev term [48]. Therefore the fourth order Lagrangian becomes in total as

$$\mathcal{L}_4 = K_1 F_{ab}^2 + K_2 \frac{(\partial_a \Phi \partial^a \Phi^*)^2}{(1 + |\Phi|^2)^4} = \frac{(2K_1 + K_2)(\partial_a \Phi^* \partial^a \Phi)^2 - 2K_1 |\partial_a \Phi \partial^a \Phi|^2}{(1 + |\Phi|^2)^4}. \quad (\text{A.6})$$

Comparing this with Eq. (A.1) we obtain relations $2K_1 + K_2 = k'_1$ and $K_2 = k'_3$.

From the isomorphism $\mathbf{CP}^1 \simeq S^2 \simeq O(3)/O(2)$, the \mathbf{CP}^1 model is equivalent to the $O(3)$ model. In order to see this equivalence, let us introduce a three vector $\mathbf{n} = (n_1, n_2, n_3)$ by

$$\begin{aligned} \mathbf{n} &= \frac{1}{\sqrt{1 + |\Phi|^2}} (1, \Phi^*) \vec{\sigma} \frac{1}{\sqrt{1 + |\Phi|^2}} \begin{pmatrix} 1 \\ \Phi \end{pmatrix} \\ &= \frac{1}{1 + |\Phi|^2} (1, \Phi^*) \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \begin{pmatrix} 1 \\ \Phi \end{pmatrix} \\ &= \left(\frac{\Phi + \Phi^*}{1 + |\Phi|^2}, \frac{-i\Phi + i\Phi^*}{1 + |\Phi|^2}, \frac{1 - |\Phi|^2}{1 + |\Phi|^2} \right) \end{aligned} \quad (\text{A.7})$$

which satisfies the constraint

$$\mathbf{n}^2 = 1. \quad (\text{A.8})$$

Conversely, Φ is the stereographic coordinate, given by

$$\Phi = \frac{n_1 + in_2}{1 + n_3} = \frac{1 - n_3}{n_1 - in_2}. \quad (\text{A.9})$$

The kinetic term becomes

$$\frac{\partial_a \Phi \partial^a \Phi^*}{(1 + |\Phi|^2)^2} = \frac{1}{2} \partial_a \mathbf{n} \cdot \partial^a \mathbf{n}, \quad (\text{A.10})$$

and the field strength can be rewritten as

$$F_{ab} = \mathbf{n} \cdot (\partial_a \mathbf{n} \times \partial_b \mathbf{n}). \quad (\text{A.11})$$

Therefore the Skyrme-Faddeev term and the other four derivative term become

$$F_{ab}^2 = (\mathbf{n} \cdot \partial_a \mathbf{n} \times \partial_b \mathbf{n})^2 = (\partial_a \mathbf{n} \times \partial_b \mathbf{n})^2, \quad (\text{A.12})$$

$$\frac{(\partial_a \Phi \partial^a \Phi^*)^2}{(1 + |\Phi|^2)^4} = \frac{1}{4} (\partial_a \mathbf{n} \cdot \partial^a \mathbf{n})^2, \quad (\text{A.13})$$

respectively. The total four derivative term \mathcal{L}_4 can be rewritten as

$$\mathcal{L}_4 = K_1 (\partial_a \mathbf{n} \times \partial_b \mathbf{n})^2 + \frac{K_2}{4} (\partial_a \mathbf{n} \cdot \partial^a \mathbf{n})^2. \quad (\text{A.14})$$

References

- [1] H. B. Nielsen and P. Olesen, Nucl. Phys. B **61**, 45 (1973).
- [2] G. 't Hooft, Nucl. Phys. B **79**, 276 (1974); A. M. Polyakov, JETP Lett. **20**, 194 (1974) [Pisma Zh. Eksp. Teor. Fiz. **20**, 430 (1974)].
- [3] See for example, A. Vilenkin and E. P. S. Shellard, “Cosmic Strings and Other Topological Defects,” (Cambridge Univ. Press, 1994); N. Manton and P. Sutcliffe, “Topological Solitons,” (Cambridge Univ. Press, 2004); N. D. Mermin, Rev. Mod. Phys. **51**, 591 (1979).
- [4] A. Chodos and E. Poppitz, Phys. Lett. B **471**, 119 (1999) [arXiv:hep-th/9909199]; M. Giovannini, H. Meyer and M. E. Shaposhnikov, Nucl. Phys. B **619**, 615 (2001) [arXiv:hep-th/0104118].
- [5] A. Hanany and D. Tong, JHEP **0307**, 037 (2003) [arXiv:hep-th/0306150].
- [6] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, Nucl. Phys. B **673**, 187 (2003) [arXiv:hep-th/0307287].
- [7] A. Hanany and D. Tong, JHEP **0404**, 066 (2004) [arXiv:hep-th/0403158].
- [8] M. Shifman and A. Yung, Phys. Rev. D **70**, 045004 (2004) [arXiv:hep-th/0403149].
- [9] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. D **72**, 025011 (2005) [arXiv:hep-th/0412048].
- [10] A. Gorsky, M. Shifman and A. Yung, Phys. Rev. D **71**, 045010 (2005) [arXiv:hep-th/0412082].
- [11] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. Lett. **96**, 161601 (2006) [arXiv:hep-th/0511088]; M. Eto, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci and N. Yokoi, Phys. Rev. D **74**, 065021 (2006) [arXiv:hep-th/0607070]; M. Eto, K. Hashimoto, G. Marmorini, M. Nitta, K. Ohashi and W. Vinci, Phys. Rev. Lett. **98**, 091602

- (2007) [arXiv:hep-th/0609214]; M. Eto *et al.*, Nucl. Phys. B **780**, 161 (2007) [arXiv:hep-th/0611313].
- [12] D. Tong, arXiv:hep-th/0509216; M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, J. Phys. A **39**, R315 (2006) [arXiv:hep-th/0602170]; M. Shifman and A. Yung, Rev. Mod. Phys. **79**, 1139 (2007) [arXiv:hep-th/0703267]; D. Tong, Annals Phys. **324**, 30 (2009) [arXiv:0809.5060 [hep-th]].
- [13] A. P. Balachandran, S. Digal and T. Matsuura, Phys. Rev. D **73**, 074009 (2006) [arXiv:hep-ph/0509276].
- [14] E. Nakano, M. Nitta and T. Matsuura, Phys. Rev. D **78**, 045002 (2008) [arXiv:0708.4096 [hep-ph]]; Prog. Theor. Phys. Suppl. **174**, 254 (2008) [arXiv:0805.4539 [hep-ph]]; M. Eto and M. Nitta, Phys. Rev. D **80**, 125007 (2009) [arXiv:0907.1278 [hep-ph]].
- [15] M. Eto, E. Nakano and M. Nitta, Phys. Rev. D (in press) [arXiv:0908.4470 [hep-ph]].
- [16] D. M. Sedrakian, D. Blaschke, K. M. Shahabasyan and M. K. Shahabasyan, Astrophysics **51** 544 (2008) [arXiv:0810.3003 [hep-ph]].
- [17] M. Gell-Mann and M. Levy, Nuovo Cim. **16**, 705 (1960); S. Weinberg, Phys. Rev. **166**, 1568 (1968); S. R. Coleman, J. Wess and B. Zumino, Phys. Rev. **177**, 2239 (1969); C. G. Callan, S. R. Coleman, J. Wess and B. Zumino, Phys. Rev. **177**, 2247 (1969).
- [18] D. V. Volkov and V. P. Akulov, Phys. Lett. B **46**, 109 (1973).
- [19] T. E. Clark and S. T. Love, Phys. Rev. D **54**, 5723 (1996) [arXiv:hep-ph/9608243]; T. E. Clark, T. Lee, S. T. Love and G. H. Wu, Phys. Rev. D **57**, 5912 (1998) [arXiv:hep-ph/9712353]; T. E. Clark, S. T. Love, M. Nitta and T. ter Veldhuis, Phys. Rev. D **73**, 125006 (2006) [arXiv:hep-th/0512078].
- [20] J. Bagger and A. Galperin, Phys. Lett. B **336**, 25 (1994) [arXiv:hep-th/9406217]; Phys. Rev. D **55**, 1091 (1997) [arXiv:hep-th/9608177]; Phys. Lett. B **412**, 296 (1997) [arXiv:hep-th/9707061].

- [21] J. Hughes and J. Polchinski, Nucl. Phys. B **278**, 147 (1986); J. Hughes, J. Liu and J. Polchinski, Phys. Lett. B **180**, 370 (1986).
- [22] E. Ivanov and S. Krivonos, Phys. Lett. B **453**, 237 (1999) [Erratum-ibid. B **657**, 269 (2007)] [arXiv:hep-th/9901003]; P. C. West, JHEP **0002**, 024 (2000) [arXiv:hep-th/0001216].
- [23] T. E. Clark, M. Nitta and T. ter Veldhuis, Phys. Rev. D **67**, 085026 (2003) [arXiv:hep-th/0208184]; Phys. Rev. D **69**, 047701 (2004) [arXiv:hep-th/0209142]; Phys. Rev. D **70**, 105005 (2004) [arXiv:hep-th/0401163]; Phys. Rev. D **71**, 025017 (2005) [arXiv:hep-th/0409030]; Phys. Rev. D **70**, 125011 (2004) [arXiv:hep-th/0409151].
- [24] L. X. Liu, Phys. Rev. D **79**, 045017 (2009) [arXiv:0711.4868 [hep-th]].
- [25] T. E. Clark, S. T. Love, M. Nitta, T. ter Veldhuis and C. Xiong, Phys. Rev. D **76**, 105014 (2007) [arXiv:hep-th/0703179]; Phys. Rev. D **75**, 065028 (2007) [arXiv:hep-th/0612147].
- [26] T. E. Clark, S. T. Love, M. Nitta and T. ter Veldhuis, J. Math. Phys. **46**, 102304 (2005) [arXiv:hep-th/0501241]; Phys. Rev. D **72**, 085014 (2005) [arXiv:hep-th/0506094]; T. E. Clark and S. T. Love, Phys. Rev. D **73**, 025001 (2006) [arXiv:hep-th/0510274].
- [27] A. M. Polyakov, Nucl. Phys. B **268**, 406 (1986); “Gauge Fields and Strings,” (Harwood Academic Publishers, 1987).
- [28] T. L. Curtright, G. I. Ghandour, C. B. Thorn and C. K. Zachos, Phys. Rev. Lett. **57**, 799 (1986); T. Curtright, G. Ghandour and C. K. Zachos, Phys. Rev. D **34**, 3811 (1986).
- [29] E. A. Ivanov and V. I. Ogievetsky, Teor. Mat. Fiz. **25**, 164 (1975).
- [30] Y. Nambu, Phys. Rev. D **10**, 4262 (1974); T. Goto, Prog. Theor. Phys. **46**, 1560 (1971).

- [31] K. Higashijima, T. Kimura and M. Nitta, *Annals Phys.* **296**, 347 (2002) [arXiv:hep-th/0110216]; *Nucl. Phys. B* **645**, 438 (2002) [arXiv:hep-th/0202064].
- [32] K. Higashijima and M. Nitta, *Prog. Theor. Phys.* **105**, 243 (2001) [arXiv:hep-th/0006027]; K. Higashijima, E. Itou and M. Nitta, *Prog. Theor. Phys.* **108**, 185 (2002) [arXiv:hep-th/0203081].
- [33] K. I. Maeda and N. Turok, *Phys. Lett. B* **202**, 376 (1988); R. Gregory, *Phys. Lett. B* **206**, 199 (1988); S. M. Barr and D. Hochberg, *Phys. Rev. D* **39**, 2308 (1989); R. Gregory, D. Haws and D. Garfinkle, *Phys. Rev. D* **42**, 343 (1990); T. H. Hansson, J. Isberg, U. Lindstrom, H. Nordstrom and J. Grundberg, *Phys. Lett. B* **261**, 379 (1991); B. Boisseau and P. S. Letelier, *Phys. Rev. D* **46**, 1721 (1992); P. Orland, *Nucl. Phys. B* **428**, 221 (1994) [arXiv:hep-th/9404140]; H. Arodz, *Nucl. Phys. B* **450**, 189 (1995) [arXiv:hep-th/9503001]; M. Anderson, *Phys. Rev. D* **51**, 2863 (1995); M. R. Anderson, F. Bonjour, R. Gregory and J. Stewart, *Phys. Rev. D* **56**, 8014 (1997) [arXiv:hep-ph/9707324].
- [34] L. Ferretti, S. B. Gudnason and K. Konishi, *Nucl. Phys. B* **789**, 84 (2008) [arXiv:0706.3854 [hep-th]]; M. Eto, T. Fujimori, S. B. Gudnason, M. Nitta and K. Ohashi, *Nucl. Phys. B* **815**, 495 (2009) [arXiv:0809.2014 [hep-th]]; M. Eto *et al.*, *JHEP* **0906**, 004 (2009) [arXiv:0903.4471 [hep-th]].
- [35] M. Eto, T. Fujimori, S. B. Gudnason, K. Konishi, M. Nitta, K. Ohashi and W. Vinci, *Phys. Lett. B* **669**, 98 (2008) [arXiv:0802.1020 [hep-th]]; S. B. Gudnason, *Nucl. Phys. B* **821**, 151 (2009) [arXiv:0906.0021 [hep-th]].
- [36] K. Itoh, T. Kugo and H. Kunitomo, *Nucl. Phys. B* **263**, 295 (1986); K. Higashijima and M. Nitta, *Prog. Theor. Phys.* **103**, 635 (2000) [arXiv:hep-th/9911139]; K. Higashijima, T. Kimura, M. Nitta and M. Tsuzuki, *Prog. Theor. Phys.* **105**, 261 (2001) [arXiv:hep-th/0010272]; M. Nitta, *Nucl. Phys. B* **711**, 133 (2005) [arXiv:hep-th/0312025].
- [37] M. Shifman and A. Yung, *Phys. Rev. D* **70**, 025013 (2004) [arXiv:hep-th/0312257]; M. Eto, M. Nitta, K. Ohashi and D. Tong, *Phys. Rev. Lett.* **95**, 252003 (2005) [arXiv:hep-th/0508130]; M. Eto, T. Fujimori, M. Nitta,

- K. Ohashi and N. Sakai, Phys. Rev. D **77**, 125008 (2008) [arXiv:0802.3135 [hep-th]].
- [38] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. Lett. **93**, 161601 (2004) [arXiv:hep-th/0404198]; Phys. Rev. D **71**, 065018 (2005) [arXiv:hep-th/0405129]; Phys. Rev. D **70**, 125014 (2004) [arXiv:hep-th/0405194]; M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta and N. Sakai, Phys. Rev. D **71**, 125006 (2005) [arXiv:hep-th/0412024]; M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, N. Sakai and Y. Tachikawa, Phys. Rev. D **71**, 105009 (2005) [arXiv:hep-th/0503033].
- [39] I. A. Bandos, D. P. Sorokin, M. Tonin, P. Pasti and D. V. Volkov, Nucl. Phys. B **446**, 79 (1995) [arXiv:hep-th/9501113]; I. A. Bandos and W. Kummer, Int. J. Mod. Phys. A **14**, 4881 (1999) [arXiv:hep-th/9703099].
- [40] T. E. Clark, S. T. Love, M. Nitta, T. ter Veldhuis and C. Xiong, Phys. Lett. B **671**, 383 (2009) [arXiv:0709.4023 [hep-th]]; Phys. Rev. D **78**, 115004 (2008) [arXiv:0809.3999 [hep-ph]]; Nucl. Phys. B **810**, 97 (2009) [arXiv:0809.1083 [hep-th]].
- [41] M. Eto, M. Nitta and N. Sakai, Nucl. Phys. B **701**, 247 (2004) [arXiv:hep-th/0405161]; M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. D **73**, 125008 (2006) [arXiv:hep-th/0602289].
- [42] T. Vachaspati and A. Achucarro, Phys. Rev. D **44**, 3067 (1991); A. Achucarro and T. Vachaspati, Phys. Rept. **327**, 347 (2000) [Phys. Rept. **327**, 427 (2000)] [arXiv:hep-ph/9904229].
- [43] M. Shifman and A. Yung, Phys. Rev. D **73**, 125012 (2006) [arXiv:hep-th/0603134]; M. Eto *et al.*, Phys. Rev. D **76**, 105002 (2007) [arXiv:0704.2218 [hep-th]].
- [44] L. D. Faddeev and A. J. Niemi, Nature **387**, 58 (1997) [arXiv:hep-th/9610193].
- [45] B. M. A. Piette, B. J. Schroers and W. J. Zakrzewski, Z. Phys. C **65**, 165 (1995) [arXiv:hep-th/9406160]; B. M. A. Piette, B. J. Schroers and W. J. Zakrzewski, Nucl. Phys. B **439**, 205 (1995) [arXiv:hep-ph/9410256].

- [46] H. Gies, Phys. Rev. D **63**, 125023 (2001) [arXiv:hep-th/0102026].
- [47] L. A. Ferreira, JHEP **0905**, 001 (2009) [arXiv:0809.4303 [hep-th]]; L. A. Ferreira, N. Sawado and K. Toda, arXiv:0908.3672 [hep-th].
- [48] L. Freyhult, Nucl. Phys. B **681**, 65 (2004) [arXiv:hep-th/0310261].